In search of equivalence relations for cryptographic Boolean functions

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Outline

Vectorial Boolean functions

For n and m positive integers
Boolean functions: $F : \mathbb{F}_2^n \to \mathbb{F}_2$
Vectorial Boolean (n, m)-functions: $F : \mathbb{F}_2^n \to \mathbb{F}_2^m$

Modern applications of Boolean functions:

- reliability theory, multicriteria analysis, mathematical biology, image processing, theoretical physics, statistics;
- voting games, artificial intelligence, management science, digital electronics, propositional logic;
- algebra, projective geometry, coding theory, combinatorics, sequence design, cryptography.

On the number of Boolean functions

BF_n is the set of Boolean functions $F : \mathbb{F}_2^n \to \mathbb{F}_2$.

$$|BF_n| = 2^{2^n}$$

n	4	5	6	7	8
$\lvert BF_n\rvert$	2^{16}	2^{32}	2^{64}	2^{128}	2^{256}
$\approx$	$6 \cdot 10^4$	$4 \cdot 10^9$	10^{19}	10^{38}	10^{77}

BF_n^n is the set of vectorial Boolean functions $F : \mathbb{F}_2^n \to \mathbb{F}_2^n$.

$$|BF_n^n| = 2^{n2^n}$$

n	4	5	6	7	8
$\lvert BF_n^n\rvert$	2^{64}	2^{160}	2^{384}	2^{896}	2^{2048}

Cryptographic properties of functions

Functions used in block ciphers, S-boxes, should possess certain properties to ensure resistance of the ciphers to cryptographic attacks.

Cryptographic attacks on block ciphers and corresponding properties of S-boxes:

- Linear attack – Nonlinearity
- Differential attack – Differential uniformity
- Algebraic attack – Existence of low degree multivariate equations
- Higher order differential attack – Algebraic degree
- Interpolation attack – Univariate polynomial degree

Optimal cryptographic functions

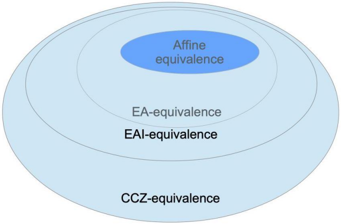
Optimal cryptographic functions

- are vectorial Boolean functions optimal for primary cryptographic criteria;
- are UNIVERSAL - they define optimal objects in several branches of mathematics and information theory (coding theory, sequence design, projective geometry, combinatorics, commutative algebra);
- are "HARD-TO-GET" - there are only a few known constructions;
- are "HARD-TO-PREDICT" - most conjectures are proven to be false.

Equivalence relations for cryptographic functions

Equivalence relations preserving main cryptographic properties

- affine equivalence - $A \circ F \circ A'$ with A and A' affine permutations;
- extended affine (EA-) equivalence $A \circ F \circ A' + A$ with A affine;
- EAI-equivalence - combination of EA-equivalence with inverses of permutations;
- CCZ-equivalence.



Importance of Equivalence Relations for Functions

Equivalence relations preserving main cryptographic properties divide the set of all functions into classes.

- Instead of checking invariant properties for all functions, it is enough to check only one function in each class.
- They can be powerful construction methods providing for each function a huge class of functions with the same invariant primary properties but with a large variety of other properties.

Outline

Univariate representation of functions

The univariate representation of $F : \mathbb{F}_{2^n} \rightarrow \mathbb{F}_{2^m}$ for $m|n$:

$$F(x) = \sum_{i=0}^{2^n-1} c_i x^i, \quad c_i \in \mathbb{F}_{2^n}.$$

The univariate degree of F is the degree of its univariate representation.

Example

$$F(x) = x^7 + \alpha x^6 + \alpha^2 x^5 + \alpha^4 x^3$$

where α is a primitive element of $\mathbb{F}_{2^3}$.

Algebraic degree of univariate function

For n a positive integer, binary expansion of an integer k, $0 \leq k < 2^n$ is

$$k = \sum_{s=0}^{n-1} 2^s k_s,$$

where k_s, $0 \leq k_s \leq 1$. Then binary weight of k:

$$w_2(k) = \sum_{s=0}^{n-1} k_s.$$

Algebraic degree of F

$$F(x) = \sum_{i=0}^{2^n-1} c_i x^i, \quad c_i \in \mathbb{F}_{2^n},$$

$$d^\circ(F) = \max_{0 \leq i < 2^n, c_i \neq 0} w_2(i).$$

Special functions

- F is linear if
$$F(x) = \sum_{i=0}^{n-1} b_i x^{2^i}.$$
- F is affine if it is a linear function plus a constant.
- F is quadratic if for some affine A
$$F(x) = \sum_{i,j=0}^{n-1} b_{ij} x^{2^i+2^j} + A(x).$$
- F is power function or monomial if $F(x) = x^d$.
- F is permutation if it is a one-to-one map.
- The inverse F^{-1} of a permutation F is s.t. $F^{-1}(F(x)) = F(F^{-1}(x)) = x$.

Trace and component functions

Trace function from $\mathbb{F}_{2^n}$ to $\mathbb{F}_{2^m}$ for $m|n$:

$$\mathrm{tr}_n^m(x) = \sum_{i=0}^{n/m-1} x^{2^{im}}.$$

Absolute trace function:

$$\mathrm{tr}_n(x) = \mathrm{tr}_n^1(x) = \sum_{i=0}^{n-1} x^{2^i}.$$

For $F : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ and $v \in \mathbb{F}_{2^n}^*$

$$\mathrm{tr}_n(vF(x))$$

is a component function of F.

Outline

Differential uniformity and APN functions

- Differential cryptanalysis of block ciphers was introduced by Biham and Shamir in 1991.
- $F : \mathbb{F}_{2^n} \rightarrow \mathbb{F}_{2^n}$ is differentially δ-uniform if

$$F(x+a)+F(x)=b, \qquad \forall a \in \mathbb{F}_{2^n}^*, \quad \forall b \in \mathbb{F}_{2^n},$$

  has at most δ solutions.
- Differential uniformity measures the resistance to differential attack [Nyberg 1993].
- F is almost perfect nonlinear (APN) if $\delta = 2$.
- APN functions are optimal for differential cryptanalysis.

First examples of APN functions [Nyberg 1993]:

- Gold function x^{2^i+1} on $\mathbb{F}_{2^n}$ with $\gcd(i,n) = 1$;
- Inverse function x^{-1} on $\mathbb{F}_{2^n}$ with n odd.

Quadratic and Power APN Functions

- $F(x) = x^d$ on $\mathbb{F}_{2^n}$, then F is APN iff for any $b \in \mathbb{F}_{2^n}$

$$(x+1)^d + x^d = b$$

- If F is quadratic then F is APN iff $F(x+a) + F(x) = F(a)$ has 2 solutions for any $a \neq 0$.

Walsh transform of an (n, n)-function F

Walsh coefficients of F:

$$\lambda_F(u, v) = \sum_{x \in \mathbb{F}_{2^n}} (-1)^{\mathrm{tr}_n(v\ F(x)) + \mathrm{tr}_n(ax)}, \quad u \in \mathbb{F}_{2^n},\ v \in \mathbb{F}_{2^n}^*$$

- Walsh spectrum of F is the set of all Walsh coefficients of F.
- The extended Walsh spectrum of F is the set of absolute values of all Walsh coefficients of F.
- F is APN iff

$$\sum_{u,v \in \mathbb{F}_{2^n}, v \neq 0} \lambda_F^4(u, v) = 2^{3n+1}(2^n - 1).$$

Nonlinearity of functions

- Linear cryptanalysis was discovered by Matsui in 1993.
- Nonlinearity measures the resistance to linear attack [Chabaud and Vaudenay 1994].

The nonlinearity of F:

$$N_F = 2^{n-1} - \frac{1}{2} \max_{u \in \mathbb{F}_{2^n}, v \in \mathbb{F}_{2^n}^*} |\lambda_F(u,v)| \leq 2^{n-1} - 2^{\frac{n-1}{2}}.$$

Functions achieving this bound are called almost bent (AB).

- AB functions are optimal for linear cryptanalysis.
- F is maximally nonlinear if n is even and $N_F = 2^{n-1} - 2^{\frac{n}{2}}$ (conjectured optimal).

Almost bent functions

- F is AB iff $\lambda_F(u, v) \in \{0, \pm 2^{\frac{n+1}{2}}\}$.
- AB functions exist only for n odd.
- If F is AB then it is APN.
- If n is odd and F is quadratic APN then F is AB.
- Algebraic degrees of AB functions are upper bounded by $\frac{n+1}{2}$ [Carlet, Charpin, Zinoviev 1998].

First example of AB functions:

- Gold functions x^{2^i+1} on $\mathbb{F}_{2^n}$ with $\gcd(i, n) = 1$, *n odd*;
- Gold APN functions with n even are not AB;
- Inverse functions are not AB.

Almost bent power functions

- Checking Walsh spectrum for power functions is sufficient for $a \in \mathbb{F}_2$ and $b \in \mathbb{F}_{2^n}^*$.
  - $F(x) = x^d$ is AB on $\mathbb{F}_{2^n}$ iff $\lambda_F(a,b) \in \{0, \pm 2^{\frac{n+1}{2}}\}$ for $a \in \mathbb{F}_2$, $b \in \mathbb{F}_{2^n}^*$.
- In case of power permutation, sufficient for $b = 1$ and all a.
  - If $F = x^d$ is a permutation, F is AB iff $\lambda_F(a,1) \in \{0, \pm 2^{\frac{n+1}{2}}\}$ for $a \in \mathbb{F}_{2^n}$.

Outline

Cyclotomic, affine, EA- and EAI- equivalences

- F and F' are affine equivalent if
$$F' = A_1 \circ F \circ A_2$$
for some affine permutations A_1 and A_2.

- F and F'' are *extended affine equivalent* (EA-equivalent) if
$$F'' = F' + A$$
where F' is affine equivalent to F and A is affine.

- F and F' are EAI-equivalent if F' is obtained from F by a sequence of applications of EA-equivalence and inverses of permutations.

- Functions x^d and $x^{d'}$ over $\mathbb{F}_{2^n}$ are cyclotomic equivalent if $d' = 2^i \cdot d \mod (2^n - 1)$ for some $0 \leq i < n$ or, $d' = 2^i / d \mod (2^n - 1)$ in case $\gcd(d, 2^n - 1) = 1$.

Invariants and relation between equivalences

- EA-equivalence and cyclotomic equivalence are particular cases of EAI-equivalence.
- APNness and ABness are preserved by EAI-equivalence.
- Algebraic degree is preserved by EA-equivalence but not by EAI-equivalence.
- Univariate degree is not preserved by any of the equivalences.
- Permutation property is preserved by cyclotomic and affine equivalences (not by EA- or EAI-equivalences).

Known AB power functions x^d on $\mathbb{F}_{2^n}$

Functions	Exponents d	Conditions on n odd
Gold (1968)	$2^i + 1$	$\gcd(i, n) = 1$, $1 \leq i < n/2$
Kasami (1971)	$2^{2i} - 2^i + 1$	$\gcd(i, n) = 1$, $2 \leq i < n/2$
Welch (conj.1968)	$2^m + 3$	$n = 2m + 1$
Niho (conjectured in 1972)	$2^m + 2^{\frac{m}{2}} - 1$, m even $2^m + 2^{\frac{3m+1}{2}} - 1$, m odd	$n = 2m + 1$

Welch and Niho cases were proven by Canteaut, Charpin, Dobbertin (2000) and Hollmann, Xiang (2001), respectively.

Known APN power functions x^d on $\mathbb{F}_{2^n}$

Functions	Exponents d	Conditions
Gold	$2^i + 1$	$\gcd(i,n) = 1$, $1 \leq i < n/2$
Kasami	$2^{2i} - 2^i + 1$	$\gcd(i,n) = 1$, $2 \leq i < n/2$
Welch	$2^m + 3$	$n = 2m + 1$
Niho	$2^m + 2^{\frac{m}{2}} - 1$, m even $2^m + 2^{\frac{3m+1}{2}} - 1$, m odd	$n = 2m + 1$
Inverse	$2^{n-1} - 1$	$n = 2m + 1$
Dobbertin	$2^{4m} + 2^{3m} + 2^{2m} + 2^m - 1$	$n = 5m$

- Power APN functions are permutations for n odd and 3-to-1 for n even [Dobbertin 1999].
- This list is up to cyclotomic equivalence and is conjectured complete [Dobbertin 1999].
- For n even the Inverse function is differentially 4-uniform and maximally nonlinear and is used as S-box in AES with $n = 8$.

Open problems in the beginning of 2000

- All known APN functions were power functions up to EA-equivalence.
- Power APN functions are permutations for n odd and 3-to-1 for n even.

Open problems:

1. Existence of APN polynomials (EA-)inequivalent to power functions.
2. Existence of APN permutations over $\mathbb{F}_{2^n}$ for n even.

CCZ-equivalence

The *graph of a function* $F : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ is the set

$$G_F = \{(x, F(x)) : x \in \mathbb{F}_{2^n}\}.$$

F and F' are CCZ-equivalent if $\mathcal{L}(G_F) = G_{F'}$ for some affine permutation $\mathcal{L}$ of $\mathbb{F}_{2^n} \times \mathbb{F}_{2^n}$ [Carlet, Charpin, Zinoviev 1998].

CCZ-equivalence

- preserves differential uniformity, nonlinearity, extended Walsh spectrum and resistance to algebraic attack.
- is more general than EAI-equivalence [B., Carlet, Pott 2005].
- was used to disprove two conjectures of 1998:
  - On nonexistence of AB functions EA-inequivalent to any permutation [disproved by B., Carlet, Pott 2005];
  - On nonexistence of APN permutations for n even [disproved for $n = 6$ by Dillon et al. 2009].

CCZ-Equivalence Formula

Let $\mathcal{L}$ be a affine permutation of $\mathbb{F}_{2^n}^2$ such that $\mathcal{L}(G_F) = G_{F'}$.

$\mathcal{L}(x, y) = (L_1(x, y), L_2(x, y))$ for some affine $L_1, L_2 : \mathbb{F}_{2^n}^2 \to \mathbb{F}_{2^n}$.

Then $\mathcal{L}(x, F(x)) = (F_1(x), F_2(x))$, where

$$F_1(x) = L_1(x, F(x)),$$

$$F_2(x) = L_2(x, F(x)),$$

and

$$\mathcal{L}(G_F) = \{(F_1(x), F_2(x)) : x \in \mathbb{F}_{2^n}\}.$$

$\mathcal{L}(G_F)$ is the graph of a function iff F_1 is a permutation.

Then, $F' = F_2 \circ F_1^{-1}$ and $\mathcal{L}(G_F) = G_{F'}$.

$$L_i(x, y) = A_{i1}(x) + A_{i2}(y)$$

for some affine $A_{ij} : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$, $i, j \in \{0, 1\}$.

Outline

Construction of CCZ-eq. but EAI-ineq. F and F'

1. Find a *permutation* $L_1(x, F(x)) = A_1 \circ F(x) + A_2(x)$ where $A_1, A_2 \neq 0$ are linear (necessary but not sufficient).
   - F' is EA-equivalent to F or to F^{-1} (if it exists) iff there exists a linear permutation $\mathcal{L} = (L_1, L_2)$ such that $\mathcal{L}(G_F) = G_{F'}$ and $L_1(x, y) = L(x)$ or $L_1(x, y) = L(y)$.
2. Then linear function $L_2(x, y) = A_3(y) + A_4(x)$ such that

$$\begin{aligned} A_1(y) + A_2(x) &= 0 \\ A_3(y) + A_4(x) &= 0 \end{aligned}$$

   has only $(0, 0)$ solution, always exist.

- To construct a permutation F' both $L_1(x, F(x))$ and $L_2(x, F(x))$ must be permutations.

CCZ-equiv. is more general than EAI-equiv.

Example: APN maps $F(x) = x^{2^i+1}$, $\gcd(i, n) = 1$, over $\mathbb{F}_{2^n}$ and $F'(x) = x^{2^i+1} + \left(x^{2^i} + x + \mathrm{tr}_n(1) + 1\right)\mathrm{tr}_n\left(x^{2^i+1} + x\,\mathrm{tr}_n(1)\right)$ are CCZ-equivalent but EAI-inequivalent.

Take for n odd $\mathcal{L}(x, y) = (L_1(x), L_2(x)) = \left(x+\mathrm{tr}_n(x)+\mathrm{tr}_n(y), y+\mathrm{tr}_n(y)+\mathrm{tr}_n(x)\right)$ and for n even $\mathcal{L}(x, y) = (L_1, L_2)(x, y) = (x + \mathrm{tr}_n(y), y)$.

For n odd F' is AB and is EA-inequivalent to permutations. This disproved the conjecture from 1998 that every AB function is EA-equivalent to permutation.

First classes of APN and AB maps EAI-inequivalent to monomials

APN functions CCZ-equivalent to Gold functions and EAI-inequivalent to power functions on $\mathbb{F}_{2^n}$; they are AB for n odd [B., Carlet, Pott 2005].

Functions	Conditions
$x^{2^i+1}+(x^{2^i}+x+\mathrm{tr}_n(1)+1)\mathrm{tr}_n(x^{2^i+1}+x\,\mathrm{tr}_n(1))$	$n\geq 4$ $\gcd(i,n)=1$
$[x+\mathrm{tr}_n^3(x^{2(2^i+1)}+x^{4(2^i+1)})+\mathrm{tr}_n(x)\mathrm{tr}_n^3(x^{2^i+1}+x^{2^{2i}(2^i+1)})]^{2^i+1}$	$6\|n$ $\gcd(i,n)=1$
$x^{2^i+1}+\mathrm{tr}_n^m(x^{2^i+1})+x^{2^i}\mathrm{tr}_n^m(x)+x\,\mathrm{tr}_n^m(x)^{2^i}$ $+[\mathrm{tr}_n^m(x)^{2^i+1}+\mathrm{tr}_n^m(x^{2^i+1})+\mathrm{tr}_n^m(x)]^{\frac{1}{2^i+1}}(x^{2^i}+\mathrm{tr}_n^m(x)^{2^i}+1)$ $+[\mathrm{tr}_n^m(x)^{2^i+1}+\mathrm{tr}_n^m(x^{2^i+1})+\mathrm{tr}_n^m(x)]^{\frac{2^i}{2^i+1}}(x+\mathrm{tr}_n^m(x))$	$m\neq n$ n odd $m\|n$ $\gcd(i,n)=1$

CCZ- and EA-classification of all functions for $n \leq 4$

Brinkmann 2019: For $n \leq 3$ for all functions over $\mathbb{F}_{2^n}$ CCZ-class=EA-class.

- $n = 1$ - $4 = 2^2$ functions: 1 CCZ-class;
  - it is affine functions, the class contains bijection;
- $n = 2$ - $256 = 2^8$ functions: 2 CCZ-classes;
  - one is affine, contains bijection;
  - another is quadratic, has no bijections.
- $n = 3$ - $16777216 = 2^{24}$ functions: 7 CCZ-classes;
  - one affine, contains bijection;
  - 3 of them are quadratic, contain bijections;
  - 3 of them are cubic, have no bijections.
- $n = 4$ - $18446744073709551616 = 2^{64}$ functions: 4713 EA-classes;
  - 194 contain bijections;
  - for 4151 CCZ-class=EA-class;
  - some CCZ-classes can contain several EA-classes containing permutations.

Outline

First APN and AB classes CCZ-ineq. to monomials

Let s, k, p be positive integers such that $n = pk$, $p = 3, 4$, $\gcd(k, p) = \gcd(s, pk) = 1$ and α primitive in $\mathbb{F}_{2^n}^*$.

$$x^{2^s+1} + \alpha^{2^k-1} x^{2^{-k}+2^{k+s}}$$

is quadratic APN on $\mathbb{F}_{2^n}$. If n is odd then this function is an AB permutation [B., Carlet, Leander 2006-2008].

This family

- disproved the conjecture from 1998 on nonexistence of quadratic AB functions inequivalent to Gold functions.

Known APN families CCZ-ineq. to power functions

N°	Functions	Conditions
C1-C2	$x^{2^s+1} + u^{2^k-1} x^{2^{ik}+2^{mk+s}}$	$n = pk, \gcd(k,3) = \gcd(s,3k) = 1, p \in \{3,4\}$, $i = sk \bmod p, m = p - i, n \geq 12, u$ primitive in $\mathbb{F}_{2^n}^*$
C3	$sx^{q+1} + x^{2^i+1} + x^{q(2^i+1)} + cx^{2^i q+1} + c^q x^{2^i+q}$	$q = 2^m, n = 2m, gcd(i,m) = 1, c \in \mathbb{F}_{2^n}, s \in \mathbb{F}_{2^n} \setminus \mathbb{F}_q$, $X^{2^i+1} + cX^{2^i} + c^q X + 1$ has no solution x s.t. $x^{q+1} = 1$
C4	$x^3 + a^{-1}\mathrm{Tr}_n(a^3 x^9)$	$a \neq 0$
C5	$x^3 + a^{-1}\mathrm{Tr}_n^3(a^3 x^9 + a^6 x^{18})$	$3 \mid n$, $a \neq 0$
C6	$x^3 + a^{-1}\mathrm{Tr}_n^3(a^6 x^{18} + a^{12} x^{36})$	$3 \mid n, a \neq 0$
C7-C9	$ux^{2^s+1} + u^{2^k} x^{2^{-k}+2^{k+s}} + vx^{2^{-k}+1} + wu^{2^k+1} x^{2^s+2^{k+s}}$	$n = 3k, \gcd(k,3) = \gcd(s,3k) = 1, v, w \in \mathbb{F}_{2^k}$, $vw \neq 1, 3 \mid (k+s), u$ primitive in $\mathbb{F}_{2^n}^*$
C10	$(x + x^{2^m})^{2^k+1} + u'(ux + u^{2^m} x^{2^m})^{(2^k+1)2^i} + u(x + x^{2^m})(ux + u^{2^m} x^{2^m})$	$n = 2m, m \geqslant 2$ even, $\gcd(k,m) = 1$ and $i \geqslant 2$ even, u primitive in $\mathbb{F}_{2^n}^*, u' \in \mathbb{F}_{2^m}$ not a cube
C11	$L(x)^{2^i} x + L(x) x^{2^i}$	
C12	$ut(x)(x^q + x) + t(x)^{2^{2i}+2^{3i}} + at(x)^{2^{2i}} (x^q + x)^{2^i} + b(x^q + x)^{2^i+1}$	$n = 2m, q = 2^m, \gcd(m,i) = 1, t(x) = u^q x + x^q u$, $X^{2^i+1} + aX + b$ has no solution over $\mathbb{F}_{2^m}$
C13	$x^3 + a(x^{2^i+1})^{2^k} + bx^{3 \cdot 2^m} + c(x^{2^{i+m}+2^m})^{2^k}$	$n = 2m = 10, (a,b,c) = (\beta,1,0,0), i = 3, k = 2, \beta$ primitive in $\mathbb{F}_{2^2}$
		$n = 2m, m$ odd, $3 \nmid m, (a,b,c) = (\beta, \beta^2, 1), \beta$ primitive in $\mathbb{F}_{2^2}$, $i \in \{m-2, m, 2m-1, (m-2)^{-1} \mod n\}$

- All are quadratic. For *n* odd they are AB otherwise have optimal nonlinearity.
- In general, these families are pairwise CCZ-inequivalent [B., Calderini, Villa, 2020].

Representatives of APN polynomial families $n \leq 11$

Dimension	Functions	Equivalent to
6	$x^{24}+ax^{17}+a^8x^{10}+ax^9+x^3$	C3
	$ax^3+x^{17}+a^4x^{24}$	C7-C9
7	$x^3+Tr_7(x^9)$	C4
8	$x^3+x^{17}+a^{48}x^{18}+a^3x^{33}+ax^{34}+x^{48}$	C3
	$x^3+Tr_8(x^9)$	C4
	$x^3+a^{-1}Tr_8(a^3x^9)$	C4
	$a(x+x^{16})(ax+a^{16}x^{16})+a^{17}(ax+a^{16}x^{16})^{12}$	C10
	$x^9+Tr_8(x^3)$	C11
9	$x^3+Tr_9(x^9)$	C4
	$x^3+Tr_9^3(x^9+x^{18})$	C5
	$x^3+Tr_9^3(x^{18}+x^{36})$	C6
	$x^3+a^{246}x^{10}+a^{47}x^{17}+a^{181}x^{66}+a^{428}x^{129}$	C11
10	$x^6+x^{33}+a^{31}x^{192}$	C3
	$x^{33}+x^{72}+a^{31}x^{258}$	C3
	$x^3+Tr_{10}(x^9)$	C4
	$x^3+a^{-1}Tr_{10}(a^3x^9)$	C4
	$x^3+a^{341}x^9+a^{682}x^{96}+x^{288}$	C13
	$x^3+a^{341}x^{129}+a^{682}x^{96}+x^{36}$	C13
	$x^3+a^{128}x^6+a^{384}x^{12}+a^{133}x^{33}+x^{34}+a^2x^{64}+x^{65}+a^{128}x^{68}+x^{96}+a^4x^{130}+a^{260}x^{136}+a^4x^{192}+a^{136}x^{260}+a^{12}x^{384}$	C12
	$x^3+a^{920}x^6+a^{153}x^{12}+a^{925}x^{33}+x^{34}+a^{794}x^{64}+x^{65}+a^{920}x^{68}+x^{96}+a^{796}x^{130}+a^{29}x^{136}+a^{796}x^{192}+a^{928}x^{260}+a^{804}x^{384}$	C12
	$x^3+a^{788}x^6+a^{21}x^{12}+a^{793}x^{33}+x^{34}+a^{662}x^{64}+x^{65}+a^{788}x^{68}+x^{96}+a^{664}x^{130}+a^{920}x^{136}+a^{664}x^{192}+a^{796}x^{260}+a^{672}x^{384}$	C12
	$x^5+a^{576}x^{18}+a^{512}x^{20}+a^{133}x^{33}+x^{36}+a^2x^{64}+a^{514}x^{80}+x^{129}+a^{512}x^{144}+x^{160}+a^{80}x^{514}+a^{16}x^{516}+a^{18}x^{576}+a^{16}x^{640}$	C12
	$x^5+a^{477}x^{18}+a^{413}x^{20}+a^{34}x^{33}+x^{36}+a^{926}x^{64}+a^{415}x^{80}+x^{129}+a^{413}x^{144}+x^{160}+a^{1004}x^{514}+a^{940}x^{516}+a^{942}x^{576}+a^{940}x^{640}$	C12
	$x^5+a^{81}x^{18}+a^{17}x^{20}+a^{661}x^{33}+x^{36}+a^{530}x^{64}+a^{19}x^{80}+x^{129}+a^{17}x^{144}+x^{160}+a^{608}x^{514}+a^{544}x^{516}+a^{546}x^{576}+a^{544}x^{640}$	C12
11	$x^3+Tr_{11}(x^9)$	C4

Infinite families are identified for

- only 3 out of 13 quadratic APN functions of $\mathbb{F}_{2^6}$;
- only 4 out of 488 quadratic APN of $\mathbb{F}_{2^7}$;
- only 7 out of more than 21 000 quadratic APN of $\mathbb{F}_{2^8}$.

APN Polynomial CCZ-Ineq. to Monomials and Quadratics

Only one known example of APN polynomial CCZ-inequivalent to quadratics and to power functions for n=6:

$$x^3 + c^{17}(x^{17} + x^{18} + x^{20} + x^{24}) + \\ c^{14}\left(\mathrm{tr}_6(c^{52}x^3 + c^6x^5 + c^{19}x^7 + c^{28}x^{11} + c^2x^{13}) + \\ \mathrm{tr}_3(c^{18}x^9) + x^{21} + x^{42}\right)$$

where c is some primitive element of $\mathbb{F}_{2^6}$ [Brinkmann, Leander 2008, Edel et al. 2008].

- No infinite families known.
- No AB examples known.

Outline

CCZ- and EA- classification of APN functions

Brinkmann, Leander 2008:

CCZ-classification finished for:

- APN functions with $n \leq 5$ (there are only power functions).

EA-classification is finished for:

- APN functions with $n \leq 5$ (there are only power functions and the ones constructed by CCZ-equivalence in 2005).

There are some partial results for

- CCZ-equivalence of quadratic APN for $n = 7, 8$ by Yu et al. 2013, Leander et al 2021, etc.;
- EA-classification of APN functions for $n \geq 6$ by Calderini 2019;
- quadratic APN functions with coefficients in $\mathbb{F}_2$ for $n \leq 9$ by B., Kaleyski, Li, Yu 2020.

Relation between equivalences for APN monomials

- Two power functions are CCZ-equivalent iff they are cyclotomic equivalent [Dempwolff 2018, Yoshiara, 2018].
- For non-quadratic power APN with $n \leq 9$ CCZ- and EAI-equivalences coincide.
  Conjectured the same for all n [B., Calderini, Villa, 2020].

Gold function x^{2^i+1}

- CCZ-class$\neq$EAI-class.

Inverse function x^{-1} [Kolsch 2021]:

- CCZ-class$=$EA-class;
- has one affine class of permutations.

Relation between equivalences for quadratic APN

- Two quadratic APN functions are CCZ-equivalent iff they are EA-equivalent [Yoshiara 2017].
- For quadratic APN functions CCZ-equivalence is more general than EAI-equivalence [B., Carlet, Leander 2009].
- For $n = 6$, the 13 quadratic APN maps have from 3 to 91 EA-classes (algebraic degree from 2 to 4) [Calderini, 2020];

Relation between equivalences for non-quadratic APN

- For non-power non-quadratic APN functions CCZ-equivalence is more general than EAI-equivalence [B., Calderini, Villa, 2020].

- For the only known APN function ($n = 6$) CCZ-inequivalent to both quadratics and to monomials, the CCZ-class contains 25 EA-equivalence classes and does not contain permutations [Calderini 2020].

Outline

Application to crooked functions

- Every quadratic AB permutation with $F(0) = 0$ is crooked.
- Crookedness is preserved only by affine equivalence.

Known crooked functions over $\mathbb{F}_{2^n}$.

Functions	Exponents d	Conditions
Gold (1968)	x^{2^i+1}	n odd
AB binomials (2006)	$x^{2^s+1} + \alpha^{2^k-1}x^{2^{-k}+2^{k+s}}$	$n = 3k$ odd

- Among all 488 quadratic AB functions with $n = 7$, only Gold maps are CCZ-equivalent to permutations.
- For $n = 9$ two new crooked functions have been found by Beierle and Leander in 2022.

Big APN problem

Do APN permutations exist for n even?

Negative results:

- no for quadratics [Nyberg 1993],
- no for $F \in \mathbb{F}_{2^4}[x]$ if $n/2$ is even [Hou 2004],
- no for $F \in \mathbb{F}_{2^{n/2}}[x]$ [Hou 2004].

CCZ-construction of APN permutation for $n = 6$

The only known APN permutation for n even [Dillon et al 2009]:

- Applying CCZ-equivalence to quadratic APN on $\mathbb{F}_{2^n}$ with $n = 6$ and c primitive

$$P(x) = x^3 + x^{10} + cx^{24}$$

obtain a nonquadratic APN permutation

$$c^{25}x^{57} + c^{30}x^{56} + c^{32}x^{50} + c^{37}x^{49} + c^{23}x^{48} + c^{39}x^{43} + c^{44}x^{42} + c^{4}x^{41} + c^{18}x^{40} + c^{46}x^{36} + c^{51}x^{35} + c^{52}x^{34} + c^{18}x^{33} + c^{56}x^{32} + c^{53}x^{29} + c^{30}x^{28} + cx^{25} + c^{58}x^{24} + c^{60}x^{22} + c^{37}x^{21} + c^{51}x^{20} + cx^{18} + c^{2}x^{17} + c^{4}x^{15} + c^{44}x^{14} + c^{32}x^{13} + c^{18}x^{12} + cx^{11} + c^{9}x^{10} + c^{17}x^{8} + c^{51}x^{7} + c^{17}x^{6} + c^{18}x^{5} + x^{4} + c^{16}x^{3} + c^{13}x$$

Used in 2013 by Bogdanov et al. in design of Fides lightweight authenticated cipher.

CCZ-construction of infinite family of APN permutation for n even?

- The quadratic APN function $P(x) = x^3 + x^{10} + cx^{24}$ admits a "butterfly" structure leading to differentially 4-uniform permutations over $\mathbb{F}_{2^n}$ with n even but not divisible by 4 [Perrin, Udovenko, Beryukov 2016].
- P is a part of a family of quadratic APN trinomials with n divisible by 3.
  - CCZ-equivalence application for construction of permutations still to be studied.
- CCZ-equivalence class of P consists of 13 EA-equivalence classes [Calderini 2020].
  - two of EA-equivalence classes contain permutations;
  - 4 affine equivalence classes contain permutations which can be represented as P_1, P_1^{-1}, P_2, P_3.

Outline

Equivalence of indicator of the graphs of functions

The indicator of the graph G_F of $F : \mathbb{F}_2^n \to \mathbb{F}_2^m$:

$$1_{G_F}(x,y) = \begin{cases} 1 & \text{if} \quad y = F(x) \\ 0 & \text{otherwise} \end{cases} .$$

- F and F' are CCZ-equivalent iff $1_{G_{F'}} = 1_{G_F} \circ L$ for some affine permutation L.
- F and F' are CCZ-equivalent iff 1_{G_F} and $1_{G_{F'}}$ are CCZ-equivalent [B., Carlet 2010].

Currently CCZ-equivalence is the most general known equivalence relation preserving APN property.

Application to commutative semifields

$\mathbb{S} = (S, +, \star)$ is a commutative semifield if all axioms of finite fields hold except associativity for multiplication.

- $\mathbb{S} = (S, +, \star)$ is considered as $\mathbb{S} = (\mathbb{F}_{p^n}, +, \star)$.
- $F : \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$ is planar (p odd) if

$$F(x+a) - F(x), \qquad \forall a \in \mathbb{F}_{p^n}^*,$$

  are permutations.
- There is one-to-one correspondence between quadratic planar functions and commutative semifields.

The only previously known infinite classes of commutative semifields defined for all odd primes p were Dickson (1906) and Albert (1952) semifields.

Some of the classes of APN polynomials were used as patterns for constructions of new such classes of semifields
[B., Helleseth 2007; Zha et al 2009; Bierbrauer 2010, etc.]

Yet another equivalence?

- Isotopisms of commutative semifields induces isotopic equivalence of quadratic planar functions more general than CCZ-equivalence [B., Helleseth 2007].
- If quadratic planar functions F and F' are isotopic equivalent then F' is EA-equivalent to

$$F(x + L(x)) - F(x) - F(L(x))$$

for some linear permutation L [B., Calderini, Carlet, Coulter, Villa 2018].

- Isotopic equivalence for APN functions?

Isotopic construction

Isotopic construction of APN functions:

$$F(x + L(x)) - F(x) - F(L(x))$$

where L is linear and F is APN.

It is not equivalence but a powerful construction method for APN functions:

- a new infinite family of quadratic APN functions;
- for $n = 6$, starting with any quadratic APN it is possible to construct all the other quadratic APNs.

Isotopic construction for planar functions [B., Calderini, Carlet, Coulter, Villa 2021].

Outline

Conclusion

- Optimal cryptographic functions - AB and APN functions
  - their infinite classes and special cases.
- Different equivalence relations preserving APNness and ABness;
  - relation between these equivalences;
  - application for construction of different types of APN and AB functions, in particular, permutations;
  - classification results with respect to these equivalences.
- Potential possibilities for a new equivalence.

Outline

Open problems

- New equivalence relations preserving differential and linear properties.
  - "isotopic" equivalence?
- Construction of an infinite family of APN permutations for n even
  - applying CCZ-equivalence to known quadratic APN family.
- Application of CCZ-equivalence in cryptanalysis.
- Classification of APN functions:
  - new families of power functions;
  - new families of APN and AB polynomials CCZ-inequivalent to quadratics;
  - classification over specific fields $\mathbb{F}_{2^n}$ with $n \geq 6$.